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Testing Backtesting: an Evaluatiion of the Basle Guidelines
for Backtesting Internal Risk Management Models of Banks

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Testing Backtesting: an Evaluation of the Basle Guidelines for Backtesting Internal Risk Management Models of Banks

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Internal risk management models and downside-risk measures such as Value-at-Risk (VaR) play an important role in contemporary banking practice. VaR measures the maximum loss born by a bank or other financial institution over a certain time period and given a certain level of confidence. Following the Basle guidelines for bank supervision, many supervisory institutions require banks to use such models and to report VaR measures on a regular basis. Capital requirements for the bank are then directly related to its reported VaR figure. In principle, following the Basle guidelines based on the internal models approach, banks are allowed to design their own risk management models for computing their VaR. This raises the question whether banks have any impetus to come up with correct models in the sense that the VaR predicted by the model matches the true VaR. This question is addressed in the present paper. In our model, banks assign negative utility to required capital reserves due to opportunity costs. Using a stylized representation of the Basle guidelines for backtesting internal risk models, we investigate whether banks are inclined to choose overly prudent or overly risky internal models. We check the robustness of the result by varying the planning horizon of the bank and the degree of fat-tailedness of the bank's asset and liability portfolio. It generally turns out that banks have a strong incentive to use internal models that under-estimate the true VaR of the bank's portfolio. Monetary penalties should be set substantially higher by supervisory institutions to realize a closer match between reported and actual VaR.

Keywords: risk management; Value-at-Risk; Basle guidelines for bank supervision and backtesting; capital requirements; fat-tailed distributions

JEL codes: E58; G28.



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. 1. Introduction

Financial markets have become more and more efficient over the last decades. Some causes underlying these developments are the steady decline of the number of regulatory conditions and the continuing progress in the field of information technology. Also the increased understanding of the advantages and risks associated with contemporary financial instruments like derivatives has led to efficiency gains in pricing financial products and allocating risks over the different participants in the market. These rapid developments in financial markets have also led to less favorable experiences, some of the most well-known of which include the abuse of derivatives and the lack of adequate supervision, e.g., Orange County, Metallgesellschaft, and Barings.

Spurred by such dramatic experiences and by the huge activity growth in financial markets in general, supervisory institutions have tried to come up with adequate guidelines for supervision. Part of these include the formulation of capital requirements to cover unexpected market risks run by banks and other financial institutions, see the report by the Basle Supervisory Committee (BSC, 1996b). Market risks are the risks due to (price) fluctuations in financial markets. Especially if one holds an investment portfolio that is highly leveraged (e.g., due to the use of options), market risk can become a substantial part of the firm's total risk profile. From a supervisory point of view, it is important that the market risk is managed adequately. On the one hand, this requires a good internal risk management and supervision practice, evidenced by the competence of responsible management and a functional separation between front-office and back-office. On the other hand, it requires the availability of a sufficient amount of funds to cover liquidity risks in cases of highly adverse market fluctuations. There are at least two reasons for the importance of capital reserves to cover market risk in contemporary financial markets. First, many financial contracts are settled on a frequent (e.g., daily) basis through the use of margin accounts. Consequently, adverse market movements are translated directly into financial losses at the end of the day. If fluctuations are very large, as for example in situations of market stress, these losses can become substantial. Second, if market movements suddenly become highly adverse, as in the case of a market crash, the liquidity in the market can evaporate very rapidly. As a result, unprofitable parts of the portfolio can in such cases not be liquidated in time and monetary losses have to be suffered. In order to stimulate the stability of the financial system, supervisory institutions have come up with guidelines regarding required capital reserves for financial institutions for covering these market risks.

One of the crucial elements of the Basle proposals for capital adequacy

and capital reserve requirements is that banks are allowed to use their own models for computing the required amount of capital, BSC (1996a,b). Earlier attempts by the BSC to formulate guidelines for capital requirements based on a standard approach applicable to all banks were met by much criticism by the profession. In reply, the internal models approach was formulated. The advantage of permitting banks to use their internal risk management models to compute their market risk is that expert knowledge can be incorporated quite easily. Moreover, the use of internal models allows for the inclusion of all kinds of particular details with respect to the bank's portfolio composition and trading strategy.

The market risk is quantified using the concept of Value-at-Risk (VaR). The VaR is the maximum loss that can occur during a certain period of time given a certain confidence level. In our setting, the VaR corresponds to a specific quantile of the profit/loss distribution of the bank's portfolio. For a textbook treatment on VaR, see Jorion (1997). The BSC (1996b) guidelines specify a 10-day VaR for a 99% confidence level, i.e., the maximum loss that can occur with a 99% probability in a 10-days period. The 10-day period is motivated by the fact that in cases of severe market stress, markets can become very illiquid. As mentioned earlier, unprofitable parts of the portfolio can in such cases not be timely liquidated. The internal risk management model of the bank can be used to compute the VaR. This VaR has to be reported to the supervisor on a regular, e.g., quarterly, basis. The capital reserve requirements dictated by the supervisor are then directly proportional to the reported VaR.

Capital requirements have a direct impact on the profit opportunity set of the bank. Therefore, abstracting from the bank's own incentive to hold capital reserves for risk management, the capital reserves required by the supervisor are undesirable for the bank from a pure profit point of view. Given this disincentive of banks to hold capital reserves, it is interesting to investigate whether banks have any impetus to construct adequate internal models for risk management. It might be argued on intuitive grounds given the above line of reasoning, that banks are inclined to use models that return low VaR values for reporting, see also Daniellson et al. (1997). This holds particularly if the amount of capital reserves deemed necessary by the bank's risk management falls below that required by the supervisory institution. That this is regarded as a serious problem is illustrated by the BSC (1996a) report on backtesting internal models. The BSC has designed guidelines to judge the adequacy of the bank's internal risk management model in capturing the true market risks of the bank. If there is evidence to suspect that the model is inadequate, monetary penalties are imposed. Suspicion is aroused if the actual trading profits fall below the VaR too often compared with the

required confidence level of 99%. Ultimately, a revision of the internal model can be imposed by the supervisor.

The aim of the present paper is twofold. First, we investigate whether the present BSC (1996a,b) guidelines on the use of internal models in conjunction with a backtesting procedure motivate banks to construct models that adequately capture the true market risk of the bank. Second, we consider the construction of optimal backtesting procedures that provide banks with a sufficiently strong incentive to construct adequate risk management models. These optimal procedures can then be confronted with the present BSC proposals.

The adequacy of the internal models approach as a useful tool for supervision has been debated in the literature by proponents of the pre-commitment approach, see, e.g., Kupiec and O'Brien (1995a,b, 1996, 1997). Others argue that, when used in conjunction with a backtesting procedure, the internal models approach has several advantages over the pre-commitment approach, see Gumerlock (1996). Kupiec (1995), however, argues that given the information regularly available to the supervisory institution, it is difficult to develop good statistical backtesting procedures that enable one to detect fraud models at an early stage. Although the aim of the present paper is not to resolve this controversy completely, the results can be used to assess the adequacy of the present Basle proposals in preventing excess risk-seeking behavior and systematic under-reporting of the true market risk.

Two major conclusions emerge from this paper. First, the present proposals for backtesting internal risk models are highly inadequate given the analytic framework used in the paper. The penalty set in the BSC (1996a) report for using internal models that produce substantial under-estimates of the true market risk is too low to discourage banks from using such models for reporting purposes. Second, if the supervisor minimizes the (expected) quadratic mismatch between the true and the reported VaR, the optimal penalty function turns out to be a type of step-function. If the number of actual trading losses is sufficiently small, no penalty is imposed in future periods. If the number of trading losses above the reported VaR exceeds a certain threshold, however, severe penalties are imposed.

The second conclusion of the present paper gives rise to some interesting policy implications. All results in the paper are obtained under the assumption that the bank knows the true distribution of its profits. In reality, this is not realistic, such that a more gradual penalty scheme could be appropriate to account for the uncertainty of model misspecification. But even given this relaxation, the maximum (optimal) punishment found is still substantially higher than that proposed in BSC (1996a). A practical strategy to implement these substantial penalties would be to use 'sticky' penalty schemes that af-

fect the profit opportunity set over several future periods. These seem easier to implement than schemes that only allow for a large one-time reduction in future profits.

The paper is set up as follows. In Section 2, the basic framework is laid out. A model is presented capturing the salient features of the BSC guidelines on the use of internal models and the associated backtesting procedure. Section 3 reports numerical results for an evaluation of the present Basle proposals. Section 4 describes the design of optimal backtesting procedures from the supervisory point of view. Concluding remarks and suggestions for future research are contained in Section 5.

2. Basic framework

As explained in the introduction, the basic framework for evaluating the Basle guidelines for model backtesting in the present paper is centered around a fixed portfolio of assets and liabilities. We abstract from the interaction of supervisory regulations and active asset and liability management of banks. This is a limitation of the present framework and we come back to it in the concluding section of this paper.

The internal models approach advocated by the Basle Supervisory Committee (BSC) allows banks to design their own models for risk management. The motives underlying this approach are explained more fully in, e.g., the amendment to the capital accord to incorporate market risks, BSC (199613). Given a fixed portfolio of assets and liabilities, choosing an internal model for computing VaR is tantamount to picking the VaR itself. If we wish to study the adequacy of internal models for assessing true market risks, therefore, we can focus on a comparison between the VaR chosen by a bank's management and the true VaR dictated by the market. Focusing directly on the VaR chosen by the bank's management allows for a considerable simplification of the subsequent modeling process.

The VaR reported by the bank translates directly into a number for required capital reserves for that bank. In particular, the capital requirements for market risk amount to three times the reported VaR. Capital reserves cannot be used for direct profit making purposes, and as such form an impediment to the profit objective of the bank. One can expect, therefore, that banks may be inclined (from a pure profit point of view) to choose VaR measures that are too low compared to the possible movements of the market. Partly in order to overcome this problem, the BSC (1996a) has laid out a framework for backtesting internal models using realized returns on portfolios held by banks. Given the results of Kupiec (1995), the proposed

backtesting procedure seems one of the most reliable tests available. It works approximately as follows. Over a period of one year or 250 trading days, the realized day-to-day returns on the bank's portfolio are compared to the VaR of the bank's portfolio. By counting the number of times the realized return falls below the reported VaR, the supervisor obtains an idea about the adequacy of the bank's internal model for describing the true market risks. Large numbers of violations of the reported VaR signal a model that underestimates the true market risk of the bank's portfolio. If the number of violations of reported VaR exceeds a certain threshold, the supervisor will increase the scaling factor for VaR, resulting in a higher amount of required capital reserves and a larger burden on profit opportunities. Consequently, by augmenting the internal models approach with a backtesting procedure, supervisory institutions present banks with a trade-off between present and future profit opportunities. On the one hand, reducing the reported VaR lowers the required capital reserves and, thus, increases the amount of funds available for profit making. On the other hand, reducing the reported VaR leads to a higher probability of violations of VaR by realized portfolio returns. Such violations are translated into a higher scaling factor for VaR in future periods and, thus, into increased capital requirements and reductions in the amount of funds available for direct profit making.

The BSC distinguishes three zones relating to the number of violations of reported VaR by realized trading losses. These zones are given in Table 1. In the Green zone, the number of violations is small, such that the internal model can be deemed adequate for capturing the true market risk. Consequently, no increase in the VaR scaling factor is required. In the Yellow zone, doubt arises as to the integrity and/or validity of the bank's model. This is reflected in the increase in the VaR scaling factor for capital requirements. If the number of violations is larger than or equal to 10, i.e., if the reported VaR is a 4% VaR rather than a 1% VaR, the model is judged inadequate. In that case, the scaling factor is raised to 4 and the bank is likely to be obliged to revise its internal model for risk management.

We model the above framework as follows. We assume that the bank uses the objective function

$$\min E_0 \left\{ 3 \cdot r(0) \cdot VaR_m(0) + \sum_{t=1}^{T-1} e^{-\rho t} \cdot r(t) \cdot VaR_m(t) \cdot f(t, VaR_m(t-1)) \right\} \quad (1)$$

The operator $E_0(\cdot)$ denotes the expectations operator given the available information at time 0. The minimization in (1) is done with respect to the reported VaR numbers at times $t = 0, \dots, T-1$, $VaR(t)$. This means that

Table 1

This table contains the number of violations during a 250 day period of reported 1% VaR figures by realized returns on a bank’s portfolio and the corresponding scaling factor for VaR for determining the capital reserves associated with market risk. Source: Basle Committee on Banking Supervision (1996a), Table 2.

Zone	Number of exceptions	Scaling factor
Green Zone	0	3.00
	1	3.00
	2	3.00
	3	3.00
	4	3.00
Yellow Zone	5	3.40
	6	3.50
	7	3.65
	8	3.75
	9	3.85
Red Zone	≥ 10	4.00

the bank chooses its internal model(s) such as to maximize the direct profit opportunities. The return rates $r(t)$ are either internal rates of return or market rates. So the objective function in (1) gives the present value of present and future opportunity costs due to supervisory regulations. Discounting takes place at the (net) rate $e^{\rho} - 1$. The quantity $f(t, \cdot)$ in (1) is a random variable which is explained next.

The random variable $f(t, VaR_m(t - 1))$ denotes the VaR multiplication factor and reflects the proposed policy of the BSC as explained above and in Table 1. We assume that the bank may choose its internal model at most once every year, and that the supervisory institution evaluates this model by means of the backtesting procedure every year. Taking 250 trading days in a one-year period, we follow BSC (1996a) and consider the events that daily trading losses over the past year fall below the reported VaR figures as independent drawings from a certain distribution. Given our present framework where we treat the portfolio of the bank as fixed, this means that the mentioned events are independent and identically distributed (i.i.d.) drawings from a Bernoulli distribution. Consequently, the total number of violations over a one-year period has a binomial distribution with parameters $n = 250$ and $p \in [0, 1]$. The probability p is determined by the internal model and the -true distribution governing market outcomes. We describe this by setting

$$p = P_t(\Pi(t) < -VaR_m(t)), \tag{2}$$

where $P_t(\cdot)$ denotes the true distribution function of the bank's profit $\Pi(t)$. The random variable $f(\cdot)$ now links the number of violations to the VaR scaling factor for the next period through the mapping laid out in Table 1. We assume that the random variables $f(t, \cdot)$ are independent from the (internal) return rates $r(t)$.

Given the specification for $f(t, \cdot)$ it is easy to see that lowering the reported VaR has two effects. First, there is a direct effect in the objective function (1), because the reserve requirement based on the reported VaR causes opportunity costs to the bank. Second, lowering the reported VaR increases the probability p in (2), such that the expectation of $f(\cdot)$ is increased. This results in a larger penalty for future VaR figures in the objective function.

In the next sections we address the question whether the VaR reported by the bank's management (VaR_m) is an adequate reflection of the bank's true risk number (VaR_t). Therefore, we introduce the parameters $c(t)$, which are defined through

$$VaR_m(t) = (1 + c(t))VaR_t(t). \quad (3)$$

The constants $c(t)$ are a kind of safety indices. Positive values of $c(t)$ indicate that the bank chooses to employ an internal model for risk management that over-estimates the true risks taken by the bank. Alternatively, negative values of $c(t)$ indicate that the bank uses a risk model that produces overly risky behavior from a supervisory point of view. In such cases the capital reserves for market risk held by the bank fall below the required amount of reserves based on the true market risk and the supervisory regulations.

We complete the basic framework by describing what happens if a bank enters the red zone mentioned in Table 1. According to BSC (1996a), if a bank enters the red zone closer inspection of the bank's internal risk management model by the supervisor is warranted. We model this by adding the following condition to the evolution of $c(t)$ in (3):

$$c(t) = \begin{cases} c''(t) & \text{if } f(s, VaR_m(s-1)) < 3 \text{ for all } s \leq t, \\ c^*(t) & \text{otherwise.} \end{cases} \quad (4)$$

The bank now has to optimize its objective function over $c_m(t)$ instead of $c(t)$. The values $c^*(t)$ are set by the supervisory institution and describe the consequences of entering the red zone in Table 1. We adopt the a very simple strategy for the supervisor by setting $c^*(t) \equiv 0$. So if the trading losses of a bank violate the bank's reported VaR more than 9 times during a period of 250 days, the bank's reported VaR is changed to the true VaR for all present and future periods. The supervisor can try to achieve this by close

inspection of the bank's internal model. It may seem somewhat optimistic to assume that, the supervisor will always be able to set $c^*(t) \equiv 0$. Describing the risk of a bank's portfolio requires expert insight into the operation and interaction of all the bank's financial instruments. Such expert knowledge may not always be at hand within the supervisory institution, e.g., due to time constraints. Therefore, it may well be the case that $c^*(t)$ is either slightly positive or negative. We comment on the sensitivity of our numerical results with respect to the choice of $c^*(t)$ in Section 3.

Although the present framework allows us to address several important questions related to supervision and the internal models approach advocated by the BSC, there are also several limitations. First, as mentioned before, we do not consider the interaction between supervisory regulations and active asset and liability management by banks. Active balance sheet management can be used as an additional instrument by the bank for reducing the number of future VaR violations if an increase in the VaR scaling factor becomes more likely because of realized VaR violations during the course of a year. Second, we have not modeled any credibility issues related to a large number of VaR violations. Credibility issues could play a role in the relationship between the bank and the supervisor or between the bank and its customers. Third, the present framework does not incorporate the guidelines of the BSC on auxiliary model testing, such as stress testing, see the BSC (199613). Stress tests reveal the internal model's behavior under adverse market circumstances and can in certain cases trigger a prompter reaction from the side of the supervisor concerning the model's adequacy. Finally, we have taken a simplistic pure profit point of view for the bank. Of course, apart from a profit motive every bank has a drive to manage its returns as well as its risks. This means that the bank has an impetus on its own to hold capital reserves if its market risk is high. This can be captured by imposing lower bounds on the allowable values of $c(t)$ in (3). The capital requirement deemed necessary by the bank, however, may differ considerably from the amount required by the supervisory institution. Therefore, the present framework is still useful for investigating the adequacy of bank's internal risk models from a supervisory point of view.

3. Evaluation of the present guidelines

In this section we conduct some numerical experiments in order to assess the effectiveness of the present BSC (1996a) proposals for backtesting internal risk management models. As mentioned in Section 2, the backtesting procedure uses a period of 250 trading days or approximately one trading

year. Throughout this section we only consider 1% VaR measures, as this is most in line with the BSC (1996) guidelines for bank supervision. Furthermore, we assume that the bank has a planning period of T years and that the reported VaR measure $VaR_m(t) = (1 + c(t)) VaR_t(t)$ must be kept fixed during the planning period, unless it is revised explicitly by the regulator because of an excess of VaR violations. This means that $c''(t)$ in (4) is fixed to c^m . We further simplify the problem by abstracting from fluctuations in the opportunity cost rate $r(t)$. We comment on the effect of relaxing this simplification further below. The objective function (1) can now be written as

$$\min_{c^m} \quad r \cdot VaR_t \cdot \left\{ f(0) \cdot (1 + c^m) + \sum_{t=1}^{T-1} e^{-\rho t} \cdot E_0 [(1 + c(t)) \cdot f(t, (1 + c(t)) \cdot VaR_t)] \right\} \quad (5)$$

where $r = r(t)$ denotes the (time-invariant) opportunity cost rate. We assume that $\rho = \ln(1.1)$, such that the discount rate is 10%. The sensitivity to the value of ρ is discussed further below. In order to rewrite the objective function in a more transparent way, we first compute the expected future capital reserves required by the supervisory regulations. To save on notation, $f(t)$ is used as a shorthand for $f(t, (1 + c(t)) \cdot VaR_t)$. Note that for $t \geq 1$

$$\begin{aligned} E_0 [(1 + c(t)) \cdot f(t)] &= \\ &E_0 [(1 + c(t)) \cdot f(t) | f(s) < 4 \forall s \in \{1, \dots, t-1\}] \cdot \\ &\quad P[f(s) < 4 \forall s \in \{1, \dots, t-1\}] + \\ &E_0 [(1 + c(t)) \cdot f(t) | \exists s \in \{1, \dots, t-1\} : f(s) \geq 4] \cdot \\ &\quad P[\exists s \in \{1, \dots, t-1\} : f(s) \geq 4], \end{aligned} \quad (6)$$

with $f(t)$ being i.i.d. The distribution of $f(t)$ can be derived as follows. Let $X(t), t = 1, \dots, T-1$, denote a set of i.i.d. binomial random variables with parameters $n = 250$ and $p(c(t))$ as in (2). $f(t)$ is then given by $f(t) = \tilde{f}(X(t))$, with $\tilde{f}(\cdot)$ the mapping as defined in Table 1. Define ξ_1^m as

$$\xi_1^m = \sum_{i=0}^9 \binom{n}{i} p(c^m)^i [1 - p(c^m)]^{n-i}. \quad (7)$$

We then have

$$\begin{aligned}
 \xi_t^m &= P[f(s) < 4 \forall s \in \{1, \dots, t\}] \\
 &= P[X(s) < 10 \forall s \in \{1, \dots, t\}] \\
 &= P[X(t) < 10 | X(s) < 10 \forall s \in \{1, \dots, t-1\}] \cdot \\
 &\quad P[X(s) < 10 \forall s \in \{1, \dots, t-1\}] \\
 &= \xi_1^m \cdot \xi_{t-1}^m,
 \end{aligned} \tag{8}$$

such that, $\xi_t^m = (\xi_1^m)^t$. The first term on the right-hand side of (6) can now be rewritten as

$$(\xi_1^m)^{t-1} \cdot [(1 + c^*) \cdot E_0^m + (c'' - c^*) \cdot A''], \tag{9}$$

with

$$E_0^m = \sum_{i=0}^n \binom{n}{i} J(i) \cdot p(c^m)^i [1 - p(c^m)]^{n-i}, \tag{10}$$

and

$$A'' = \sum_{i=0}^9 \binom{n}{i} f(i) \cdot p(c^m)^i [1 - p(c^m)]^{n-i}. \tag{11}$$

Note that E_0^m is a number and must not be confused with the expectations operator $E_0(\cdot)$. Analogously, the second term on the right-hand side of (6) can be rewritten as

$$[1 - (\xi_1^m)^{t-1}] \cdot [(1 + c^*) \cdot E_0^*], \tag{12}$$

with

$$E_0^* = \sum_{i=0}^n \binom{n}{i} \tilde{f}(i) \cdot p(c^*)^i [1 - p(c^*)]^{n-i}. \tag{13}$$

The objective function (5) can now be rewritten as

$$\begin{aligned}
 \min_{c^m} \quad & r \cdot VaR_t \cdot \left\{ f(0) \cdot (1 + c^m) + \sum_{t=1}^{T-1} e^{-\rho t} \cdot \right. \\
 & [(\xi_1^m)^{t-1} \cdot ((1 + c^*) \cdot E_0^m + (c'' - c^*) \cdot A'') + \\
 & \left. (1 - (\xi_1^m)^{t-1}) \cdot (1 + c^*) \cdot E_0^*] \right\}
 \end{aligned} \tag{14}$$

Note that ξ_1^m , E_0^m , and A^m depend on the value of c^m . Using straightforward algebra, (14) can be further simplified as

$$\min_{c^m} \quad r \cdot VaR_t \cdot \left\{ f(0) \cdot (1 + c^m) + \right. \\ \left. e^{-\rho} \frac{1 - (\xi_1^m e^{-\rho})^{T-1}}{1 - \xi_1^m e^{-\rho}} \cdot [(1 + c^*) \cdot (E_0^m - E_0^*) + (c^m - c^*) \cdot A^m] \right. \\ \left. e^{-\rho} \cdot (1 + c^*) \cdot E_0^* \cdot \frac{1 - e^{-\rho \cdot (T-1)}}{1 - e^{-\rho}} \right\} \quad (15)$$

It is clear from (15) that c_m has several different effects on the objective function. First, there is a direct effect by the multiplication factor c^m in several places. Second, increasing c^m leads to a decrease in the probability of VaR violations and, thus, to a decrease of ξ_1^m . Finally! a decreased probability of VaR violations leads to a lower expected VaR scaling factor for future periods, i.e., to smaller values of E_0^m and A^m .

Before we can actually compute the optimal value of c^m , we have to specify a functional form for p in (2). We assume that the profit Π on the bank's portfolio follows a Student t distribution with ν degrees of freedom. The Student distribution nests the familiar normal distribution for $\nu \rightarrow \infty$. By considering Student t distributions instead of the normal, we can study the effect of leptokurtosis on the optimal model choice of the bank. Leptokurtosis is a common phenomenon in financial markets, see, e.g., Pagan (1996) and Campbell et al. (1997). The probability of a violation of reported VaR now becomes

$$p(c) \equiv p = P_t(\Pi < -(1 + c) \cdot VaR_t), \quad (16)$$

with $p(0) = 1\%$, and where $P_t(\cdot)$ denotes a Student t distribution function with ν degrees of freedom. Define μ_Π and σ_Π to be the mean and standard deviation of Π , respectively. Furthermore, let $T_\nu(\cdot)$ be the cumulative distribution function (c.d.f.) of the standard Student t distribution with ν degrees of freedom, and let $T_\nu^{-1}(\cdot)$ denote the inverse standard c.d.f. Finally, let SR be the Sharpe-ratio of the profit random variable, i.e., $SR = \mu_\Pi / \sigma_\Pi$. Using these definitions and the fact that

$$VaR_t = -\mu_\Pi - (1 - 2\nu^{-1}) \cdot \sigma_\Pi T_\nu^{-1}(0.01),$$

$p(c)$ can be rewritten as

$$p(c) = P_t \left(\frac{\Pi - \mu_\Pi}{(1 - 2\nu^{-1})\sigma_\Pi} < \frac{-(1 + c) \cdot VaR_t - \mu_\Pi}{(1 - 2\nu^{-1})\sigma_\Pi} \right)$$

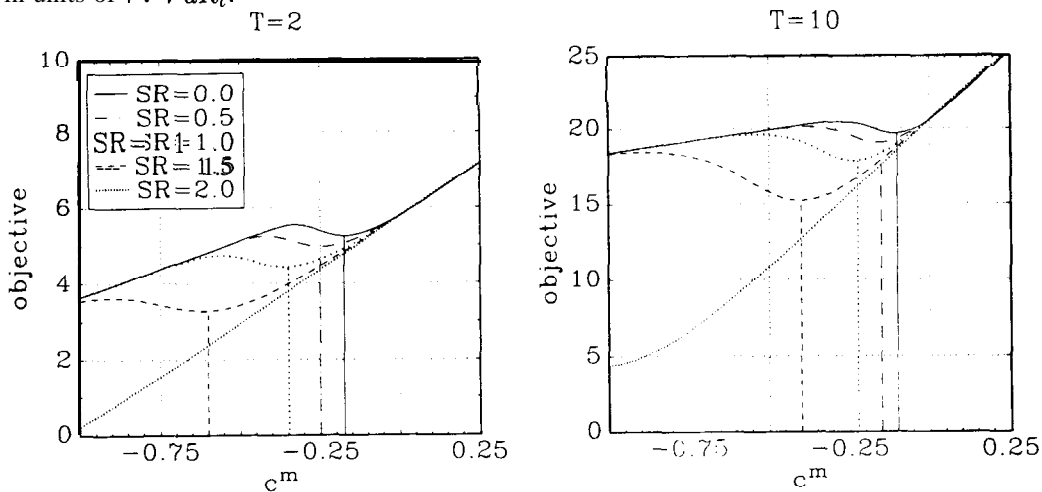
$$= \left(T_\nu (1 + c) \cdot T_\nu^{-1}(0.01) + \frac{\nu \cdot SR}{\nu - 1} \right). \quad (17)$$

It follows directly from (17) that $p(0) = 1\%$. Furthermore, (17) reveals that the probability of future VaR violations not only depends on the model chosen, i.e., on c , but also on the degree of fat-tailedness (ν) of profits and on the overall risk of the bank's portfolio through the presence of the Sharpe-ratio SR . Banks with high Sharpe-ratios profit less from choosing safe models ($c > 0$) in terms of reductions in $p(c)$ than banks with low Sharpe-ratios. This is most easily understood by considering two banks with the same value of ν , but different Sharpe-ratios. In that case, the bank with the higher Sharpe-ratio has a safer portfolio and, thus, a smaller true VaR figure. Increasing the true VaR by the same percentage amount for both banks, therefore, reduces the probability of VaR violations comparatively more for the bank with the low Sharpe-ratio, because the nominal shift in the VaR for this bank is larger than for the bank with the high Sharpe-ratio. Figure 1 presents graphs of the objective function in case $P_t(\cdot)$ is the normal distribution ($\nu \rightarrow \infty$). The objective function is plotted for different values of the planning horizon T and several values of the Sharpe-ratio SR .

Several things can be noted in Figure 1. First of all, a local minimum of the objective function is situated at $c^m < 0$ for all curves presented for which $SR < 2$. This means that banks have an incentive to choose overly risky internal risk management models. We concentrate on the local minimum for the remaining discussion and call it the *optimal* value of c^m . This is not unreasonable, given there is a natural lower bound on the values of c^m allowed by the supervisor. According to the BSC (1996) guidelines, the reported VaR based on the internal model may not fall below 50% of the VaR based on the standard BSC approach. This puts a lower bound on the allowable values of c^m . Furthermore, the bank possibly has its own lower bound for allowable values of c^m due to risk aversion motives which are not incorporated in the present framework, see the discussion in Section 2. The magnitude of the optimal c^m is substantial, indicating that it is optimal for the bank to report an under-estimate of the true VaR by 25% or more if $T = 2$ and $SR > 0.5$. For longer planning horizons, the mismatch between true and reported VaR is generally smaller, but still substantial. The increase in the optimal value of c^m for larger values of the planning horizon T is intuitively clear. If more future opportunity costs are taken into account and if the internal risk model must be chosen once and for all, then a safer risk management model will, *ceteris paribus*, result in a smaller value of the objective function. The effect is more pronounced if we set $c^* > 0$ (not shown). In that case the expected future opportunity costs of the bank increase, because the default model

Figure 1

The figure contains the objective function (15) for different values of the Sharpe-ratio (SR) and different lengths of the planning horizon (T). $100 \cdot c$ denotes the percentage increase of the reported VaR (VaR_m) with respect to the true VaR (VaR_t). At the local minimum of each curve, a vertical line indicates the c -value corresponding to this local minimum. The discount rate used for constructing the plots is 10%. The bank has to adopt the true model if it enters the red zone of Table 1, i.e., if $c^* = 0$. Without loss of generality, $r \cdot VaR_t$ is set to one, such that the vertical axis measures the value of the objective function (15) in units of $r \cdot VaR_t$.



(c^*) sets a higher VaR for reporting purposes than the true VaR. Even in this case, however, the optimal values of c^m remain negative, such that it is still profitable for the bank to report under-estimates of its true VaR to the supervisor. Extreme parameter configurations are needed (e.g., $T = 100$, $c^* = 3$) to drive the optimal value of c^m to the positive half-line.

Second, and related to the first characteristic of Figure 1, the excess degree of prudence of the internal risk model rises if the discount rate is smaller (not shown). Smaller discount rates imply that future opportunity costs are weighed more heavily in the objective function. Choosing a reported VaR below the true market risk, i.e., $c^m < 0$, causes an increase in (expected) opportunity costs during future periods. Consequently, the optimal value of VaR_m (or c^m) is decreasing in the discount rate ρ . Note that for extreme discounting $\rho \rightarrow \infty$, only the first period opportunity costs are taken into account, such that the objective function becomes monotonic in c^m .

Third note the non-monotonic shape of the objective functions in c^m . For $c^m = -1$, the reported VaR in the first period equals 0. Of course, this is an unreasonable value given the BSC lower bound on reported VaR mentioned above. By looking at $c^m = -1$, however, it is clear that for significant under-

estimates of the true VaR, the bank is (almost surely) forced in the next period to adopt the true model as an internal risk management model, i.e., to set $c(t) = c^* = 0$ for $t \geq 1$. Consequently, for extreme under-estimates of the true VaR, the objective function becomes linear in c^m with slope coefficient $r \cdot f(0) \cdot VaR_t = 3 \cdot r \cdot VaR_t$. For extreme over-estimates of the true VaR, the opposite happens. In that case future VaR violations become extremely unlikely, such that the objective function effectively collapses to

$$r \cdot VaR_t \cdot f(0) \cdot (1 + c^m) + \frac{1 - e^{-\rho \cdot (T-1)}}{1 - e^{-\rho}}, \quad (18)$$

which is again linear in c^m with a slope coefficient larger than $r \cdot f(0) \cdot VaR_t$. In between these two extremes, there is a range of values for c^m to link the two different linear segments. It is in this range that the trade-off between reductions in present opportunity costs versus an increase in expected future opportunity costs becomes really apparent through the non-monotonic behavior of the objective function.

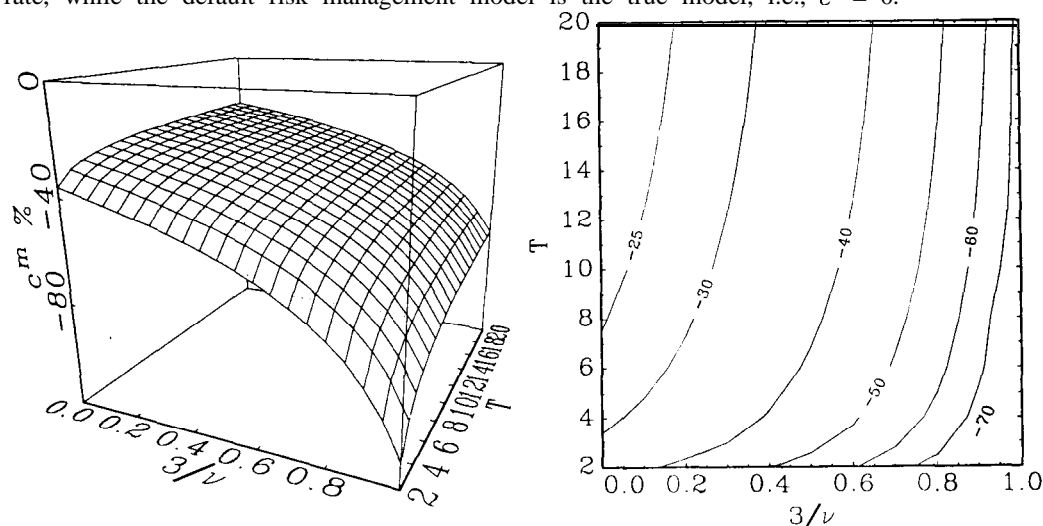
Fourth, the Sharpe-ratio has a negative impact on the optimal value of c^m in the figures presented. Banks with less risky portfolios in terms of higher Sharpe-ratios choose internal models that set a lower VaR for reporting purposes. As mentioned earlier, a bank with a high Sharpe-ratio profits less in terms of a reduction in expected future opportunity costs when the bank increases its reporting VaR. Therefore, the bank with a large Sharpe-ratio places more emphasis on reducing present instead of future opportunity costs. This results in lower values of c chosen by banks with higher Sharpe-ratios.

To conclude the discussion of Figure 1, we briefly comment on the effect of interest rate movements on the results obtained so far. If one expects increases in the interest rate $r(t)$, future opportunity are weighed more heavily in the objective function compared to present opportunity costs. Following the previous line of argument, this means that banks will be inclined to increase the degree of prudence of their internal risk models. The reverse holds if decreases in the interest, rate are expected.

We now turn to a discussion of the robustness of our results with respect to the degree of fat-tailedness of the profit distribution. The degree of fat-tailedness can be tuned by setting the parameter ν . The larger the value of ν , the more the profit distribution resembles the familiar normal distribution. Changing ν triggers several different effects. First, decreasing ν shifts the 1% quantile $T_\nu^{-1}(0.01)$ to the left. Second, for lower values of ν the Sharpe-ratio becomes more important for the effect of c^m on the probability of VaR violation $p(c^m)$. If two banks have the same Sharpe-ratio, the bank with the fatter-tailed profit distribution profits less from an increase in the prudence

Figure 2

The figure contains the (locally) optimal percentage increase of the reported VaR with respect to the true VaR, i.e., $100 \cdot c^m$. The objective function used is (15). The Sharpe-ratio used for constructing the figure is $SR = 1.0$. The optimal value of c is graphed as a function of the planning horizon T and the degree of fat-tailedness $3/\nu$. Higher values of $3/\nu$ indicate that the underlying distribution is more fat-tailed. Discounting is at a 10% rate, while the default risk management model is the true model, i.e., $c^* = 0$.



level (c^m) of the reported VaR. Third, there is an effect of ν through the c.d.f. $T_\nu(\cdot)$ used to compute $p(c^m)$ in (7). The total impact of the combination of these three effects is difficult to predict a priori. Therefore, we compute the optimal value of c^m using the objective function (13) for different values of the planning horizon T and different degrees of leptokurtosis ν . The results are presented in Figure 2. Note that we use $3/\nu$ instead of ν as a plotting variable for reasons of layout. The normal distribution now corresponds to $3/\nu = 0$, while the most fat-tailed (finite variance) distribution considered is $3/\nu = 1$, or $\nu = 3$.

Note that in order to construct Figure 2, the Sharpe-ratio is kept fixed. In effect, this means that the variance of the profit distribution is kept fixed as the degree of fat-tailedness $3/\nu$ is increased. This leads to a composite effect. First, larger values of $3/\nu$ lead to fatter tails, such that extreme profits become more likely. Second, if the variance is held fixed, larger values of $3/\nu$ lead to an increased precision ($\nu/(\sigma_\Pi \cdot (\nu - 2))$) of the Student t distribution and, thus, to a decrease in the probability of extreme profits. For more details on this, see Lucas and Klaassen (1996).

The increase in the optimal value of c^m as a function of T for fixed ν

is evident in Figure 2. Furthermore, the optimal value of c^m is decreasing in the degree of leptokurtosis $3/\nu$ for fixed T . This means that the effect of ν through a reduced effect of c^m on $p(c^m)$ governs the composite effect mentioned above. Banks with a higher value of $3/\nu$ obtain relatively less reward from raising their reported VaR above the true level ($c > 0$) in terms of a decrease in expected future opportunity costs, see (17). Consequently, for short planning horizons, these banks will be more inclined to emphasize reductions in present opportunity costs by choosing a, *ceteris paribus*, lower value of c^m . Figure 2 reveals that this effect may be so strong that the bank will even opt for reporting VaR values more than 70% below the true risk level if the profit distribution is fat-tailed, e.g., $3/\nu = 1$.

To conclude this section, we summarize the main findings. If a bank is forced by the supervisory institution to pick an internal risk model for the entire planning period: and if this model is subjected to backtesting according to the BSC (1996a) report, banks generally select overly risky internal risk management models. The effect is more pronounced for shorter planning horizons: smaller discount rates, fatter tails for the profit distribution, and higher values of the Sharpe-ratio. By contrast, if the default model upon entering the red zone overestimates the true VaR, i.e., if $c^* > 0$, relatively more prudent risk management models are chosen. Extreme parameter configurations are needed to drive the (locally) optimal value of c^m to the positive half-line.

4. Designing optimal backtesting procedures

So far, we have concentrated on the optimal choice of the bank's reporting VaR given the supervisory regulations as laid out in the BSC (1996a,b) guidelines. We now turn to a second important question. Given the bank's incentive to substantially under-estimate the true VaR for reporting purposes, what is the optimal backtesting approach for the supervisor? It is clear from the previous section that the monetary penalties as proposed by the BSC (1996a) are insufficient to guarantee a close match between reported and true VaR. We expect, therefore, that optimal backtesting procedures will set much higher monetary penalties than the ones presented in Section 2. It is the aim of the present section to quantify such optimal penalty schemes and associated backtesting procedures.

Before we can proceed with the analysis, some choices must be made regarding the objectives of the supervisor and the instruments available for achieving these objectives. Note that the bank's optimal value of c^m depends on several parameters: namely the planning horizon T , the discount rate ρ ,

the bank's Sharpe-ratio SR , and the degree of fat-tailedness of the profit distribution as characterized by the degrees of freedom parameter ν . We again abstract from fluctuations in the opportunity cost rate $r(t)$, which also have an effect on the optimal value of the bank's reported VaR. We assume that the bank has some prior ideas concerning the values of the above parameters. These ideas, possibly updated by empirical research, are summarized in the form of a (posterior) distribution function $\pi(T, \rho, SR, \nu)$. The supervisor now minimizes the objective function

$$E_{\pi}((c^m)^2), \quad (19)$$

where $E_{\pi}(\cdot)$ denotes the expectations operator with respect to the posterior distribution π . (19) states that the supervisor minimizes the (quadratic) percentage mismatch between the true VaR and the reported VaR. If this minimization has to be carried out for known values of T , ρ , SR , and ν , the posterior distribution can be chosen to have a unit mass point at the known parameter values. Alternatively, if an adequate backtesting procedure is needed for a broader range of parameter values, a non-degenerate support of the posterior distribution can be chosen.

In this section we treat ρ and T as given. The VaR mismatch is thus averaged with respect to SR and ν only. The results are remarkably stable with respect to variations in ρ and T , such that we only report the findings for $e^{\rho} - 1 = 10\%$ and $T = 10$. We assume that the remaining posterior distribution is uniform on a grid of values for (SR, ν) . We consider $\nu = 5, 10, \infty$ and $SR = \nu \cdot SR' / (\nu - 2)$, with $SR' = 0.5, 1.0, 1.3$. The limited number of combinations taken into account is motivated by two reasons. First, the optimization problem using objective function (19) is very computer intensive. A simple choice for the posterior distribution can speed up the calculations considerably. Second, the optimal solution is mainly driven by the corners of the (SR, ν) -grid considered. Therefore, we do not expect to lose much information when discarding many intermediate combinations of Sharpe-ratios and degrees of freedom parameters.

We now turn to the available instruments for minimizing the objective function (19). Given the framework laid out in Sections 2 and 3, we have as possible instruments the length of the backtesting period n , the penalty function $f(i)$, and the choice of the default model c^* . We concentrate on the first two of these. The value of c^* is zero in all computations presented below. Positive values of c^* result in a permanent increase in opportunity costs. Using the present value of these opportunity costs, positive values of c^* can be represented by high temporary penalties, i.e., by high values of $f(i)$. Some unreported experiments reveal that the qualitative results

of the present section remain unaltered if the yearly backtesting procedure ($n = 250$) is shortened to a bi-annual or a quarterly frequency. Therefore, we only present the results for $n = 250$. Note, however, that shortening the backtesting period would have adverse effects on the general statistical reliability of the backtesting procedure, see Kupiec (1995):

For the function $f(i)$, we consider the following parametric specification:

$$f(i) = \begin{cases} 3 & \text{for } 0 \leq i < 4, \\ f(4) + \alpha_2 / (1 + \exp(\alpha_1 \cdot (i - \alpha_0))) & \text{for } 4 \leq i < 10, \\ f(9) & \text{for } 10 \leq i, \end{cases} \quad (20)$$

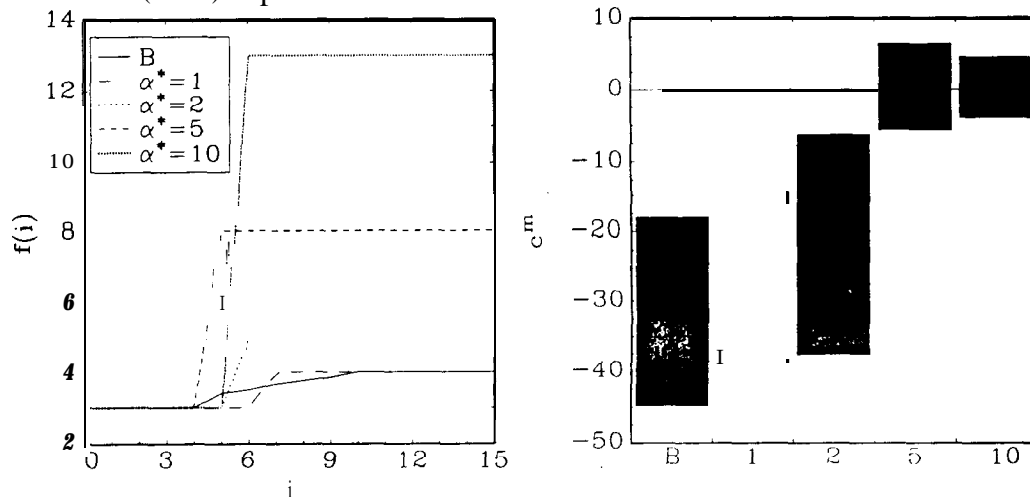
with α_0 , α_1 , and α_2 parameters that can be tuned by the supervisor. We assume that $\alpha_1 \leq 0$, such that (20) is a scaled version of the logistic distribution function. The parameter α_2 determines the maximum penalty that can be imposed, while α_0 and α_1 determine the number of VaR violations after which the VaR scaling factor starts to increase and the speed of this increase, respectively. The functional specification in (20) can fit a wide variety of penalty functions by an appropriate choice of the parameters. The sensitivity to the specific form chosen was checked by performing a similar experiment with a fully non-parametric version of the penalty function $f(i)$. The results of both experiments are very similar. Therefore, we only report the findings for the specification in (20).

The minimization problem now goes as follows. For a given set of parameters $\alpha_0, \dots, \alpha_2$, the banks set their (locally) optimal mismatch between true and reported VaR. The c^m values corresponding to the local minima of the bank's objective function (see Section 3) are squared and subsequently averaged over all values in the grid for (SR, ν) considered. This yields the objective function value (19) of the supervisor. The supervisor now minimizes this objective function with respect to $\alpha_0, \dots, \alpha_2$. For computational reasons, we impose an upper bound on the maximum penalty, $\alpha_2 \leq \alpha^*$. The results are presented in Figure 3.

It is immediately apparent from the left-hand panel in Figure 3 that the optimal form of $f(i)$ does not look at all like the penalty function proposed by the BSC (1996a). Three main differences can be seen. First, the number of violations for which no additional penalty is imposed through an increase in the VaR scaling factor is larger for the optimal penalty function if the upper bound α^* is either sufficiently large or sufficiently small. Second, while the penalty function proposed by the BSC shows a gradual increase from the base factor 3 to the maximum factor 4, the optimal $f(i)$ is a type of step function. Up to a certain threshold for VaR violations, no penalties are imposed on the bank. If the number of VaR violations exceeds this threshold, however, a

Figure 3

The left panel of the figure contains the optimal penalty function $f(i)$ for the supervisor for various values for the upper bound α^* on α_2 . For reference, the penalty function proposed by the BSC (1996a) is also presented. i denotes the number of VaR violations during the last $n = 250$ trading days. The planning period used for the figure is $T = 10$, while discounting takes place at a rate of 10%. The right panel contains the interval of optimal percentage mismatches c^m over the considered grid of values for the Sharpe-ratio SR and the degrees of freedom parameter ν . The horizontal axis in the right, panel gives the value of the upper bound α^* on α_2 , while **B** denotes the penalty function according to the BSC (1996a) report.



comparatively constant penalty is imposed. Third, the maximum increase in the scaling factor for the Basle proposals falls far below the optimal maximum penalty. In particular, the optimal maximum penalty appears to coincide with the upper bound α^* used in the computations.

The right-hand panel in Figure 3 gives insight into the maximum percentage mismatch between true and reported VaR. It appears that within the present framework, the Basle guidelines result in severe under-estimates of the true VaR. Maximum penalties should be set more than twice as high in order to drive the absolute mismatches below 6.5%.

The practical implementability of the step-type penalty function exhibited in Figure 3 warrants one cautionary remark. The derivations so far hinge on the assumption that the bank knows its true VaR, while the supervisor does not. Although it is reasonable to presuppose that the bank has a better understanding of its VaR than the supervisor, it is unrealistic to assume that the bank can estimate its VaR without error. If we relax the assumption of complete knowledge of the true VaR by the bank, a more gradual shape

of the optimal penalty function might be more appropriate in order to account for the possibility of unintentional VaR mis-specification by the bank. Alternatively, the supervisory institution could retain the step-type penalty function and put the whole burden of accounting for estimation risk on the bank. This would stimulate the banks somewhat more to design models that produce more prudent VaR numbers.

5. Concluding remarks

In this paper we have evaluated the Basle proposals for the use of internal models in conjunction with backtesting procedures, BSC (1996a,b). It turns out that the present proposals for imposing penalties on banks that violate their VaR bounds too often, are highly inadequate. The monetary penalties are too low to provide a sufficiently strong incentive to banks to design internal risk management models that produce good estimates of their true VaR. Consequently, it is profitable for banks to report under-estimates of their true VaR to the supervisory institution.

The optimal strategy for the supervisor given the above findings is to set much higher monetary penalties on an excess number of VaR violations. Moreover, the penalty function can be made much steeper than in the BSC (1996a) proposal. One obvious way to enhance the practical implementability of the higher penalty scheme is to use sticky penalties: instead of a one-time increase in the VaR scaling factor of the bank, a multi-period increase might prove more appropriate.

Several interesting questions for future research remain. For example, it is interesting to investigate the effect of active asset and liability management (ALM) from the side of the bank on the optimal supervisory policies. If banks are able through active ALM to limit the number of VaR violations if a certain number of VaR violations has already occurred during the supervisory period, then we expect even more severe under-estimates of the true VaR for reporting purposes. Modeling the internal ALM process of the bank, however, is far from trivial, and more research is needed to design an adequate and tractable framework. It would also be interesting to add uncertainty to the model in the form of an unknown value for the true VaR. Although this would highly complicate matters within the present framework, it seems more realistic that neither the bank itself nor the supervisor can come up with a faultless estimate of the true VaR.

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